Period 1, April 22, 2024

Accumulation Free Response Homework

Time: 30 mins

Directions: Answer each question, showing work as if this were an AP FRQ question. 9 points each.

Graphing Calculator Permitted

$21 \cdot 2 = 42$

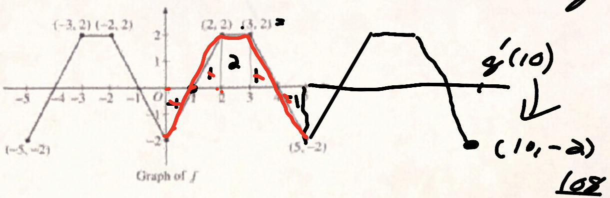
$\times 2$

44

$g(105) = 21 \cdot 2 = 42$

$\frac{105}{5} = 21$

$g'(108) = 2$



Graph of f

$\frac{108}{5} = 21\frac{3}{5}$

+9

1. The graph of the function f shown above consists of six line segments. Let g be the function given by $g(x) = \int_0^x f(t)\,dt$.

(a) Find g(4), g'(4), and g''(4).

$g(4) = \int_0^4 f(t)\,dt = -\frac{2(1)}{2} + \frac{2(1)}{2} + 2(1) + \frac{2(1)}{2} = 2+1 = 3$ ✓

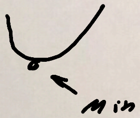
$g'(4) = f(4) ✓ = 0$ ✓

$g''(4) = f'(4) ✓ = \frac{0-2}{4-3} = -\frac{2}{1} = -2$ ✓

(b) Does g have a relative minimum, a relative maximum, or neither at $x = 1$? Justify your answer.

$g'(x) = f(x) = 0$

g has a relative minimum ✓ at x = 1 b/c $g'(x) = f(x)$ changes from negative ✓ to positive.



$g(x) = \int_0^x F(T)\,dT$

$g'(x) = F(x)$

$F(1) = 0$

aT F(1), iT goes From − To +

g'(1) " " " " " "

$\int_5^{10} F(T)\,dT = G(10) - G(5) = 4 - 2 = 2$

(c) Suppose that f is defined for all real numbers x and is periodic with a period of length 5. The graph above shows two period of f. Given that g(5) = 2, find g(10) and write an equation for the line tangent to the graph of g at $x = 108$.

$g(108) = 44$

$g(10) = 2\int_0^5 f(x)\,dx = 2\,g(5) = 2(2) = 4$

$g(108) = \int_0^{105} f(x)\,dx + \int_0^3 f(x)\,dx = \frac{105}{5}(g(5)) + g(3) = 21(2) + 2 = 42 + 2 = 44$ ✓

$g'(108) = f(108) = f(3) = 2$

$y - y_1 = m(x - x_1)$ +2 $y = mx + b$

the eq. of the tangent line at x = 108 is $y - 44 = 2(x - 108)$

+44 +44

$Y = 2(x - 108) + 44$

6/6

Graphing Calculator NOT Permitted

2. A cubic polynomial function f is defined by $f(x) = 4x^3 + ax^2 + bx + k$ where a, b, and k are constants. The function f has a local minimum at $x = -1$, and the graph of f has a point of inflection at $x = -2$.

(a) Find the values of a and b.

$f'(-1) = 0$

$f'(x) = 12x^2 + 2ax + b$ ✓

$12(-1)^2 + 2a(-1) + b = 0$

$12 - 2a + b = 0$

$12 - 2(24) + b = 0$

$12 - 48 + b = 0$

$-36 + b = 0$

$b = 36$ ✓

$f''(-2) = 0$

$f''(x) = 24x + 2a$ ✓

$24(-2) + 2a = 0$

$= -48 + 2a = 0$

$2a = 48$ ✓

$a = 24$

(b) If $\int_0^1 f(x)\,dx = 32$, what is the value of k?

$\int_0^1 f(x)\,dx = 32$

$\int_0^1 4x^3 + 24x^2 + 36x + K = 32$

$x^4 + 8x^3 + 18x^2 + Kx \Big|_0^1 = 32$ ✓

$(1 + 8 + 18 + k) - (0 + 0 + 0 + 0) = 32$

$27 + K = 32$ ✓

$K = 5$ ✓

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(2) = 3$.

2 3

1. Consider the differential equation $\frac{dy}{dx} = \frac{y^2}{x-1}$

$$\cancel{\frac{dx}{y^2}} \cdot \frac{dy}{\cancel{dx}} = \frac{\cancel{y^2}}{x-1} \cdot \frac{dx}{\cancel{y^2}}$$

$$\int \frac{1}{y^2}\,dy = \int \frac{1}{x-1}\,dx$$

$$\int y^{-2}\,dy = \int \frac{1}{x-1}\,dx$$

$$\frac{1}{-1} y^{-2+1=-1} + C_2 = \qquad u = x-1$$

$$du = dx$$

$$-y^{-1} + c_2 \qquad \int \frac{1}{u}\,du$$

$$-\frac{1}{y} + \cancel{c_2} - c_2 = \ln u + c_1 - c_2$$

$$-\frac{1}{y} = \ln|x-1| + c_3$$

$$-\frac{1}{3} = \ln|2-1| + c_3$$

$$\ln|1|$$

$$-\frac{1}{3} = 0 + c_3$$

$$-\frac{1}{3} = c_3$$

$$\cancel{y} \cdot -\frac{1}{\cancel{y}} = \left(\ln|x-1| - \frac{1}{3}\right) \cdot \cancel{y}$$

$$\frac{1}{-(\ln|x-1| - \frac{1}{3})} = \frac{-\cancel{(\ln|x-1| - \frac{1}{3})} \cdot y}{-\cancel{(\ln|x-1| - \frac{1}{3})}}$$

$$\frac{1}{(-\ln|x-1| + \frac{1}{3})} = y$$

2. Consider the differential equation $\frac{dy}{dx} = y^2\cos(2 - 2\pi x)$

$g' = y^2\cos(2-2\pi x)$

$g'(\frac{1}{\pi}) = (-\frac{1}{\pi})^2 \cos(2 - 2\cancel{\pi}\cdot\frac{1}{\cancel{\pi}})$

(a) Let $y = g(x)$ be the particular solution to the differential equation with the initial condition $g\left(\frac{1}{\pi}\right) = -\frac{1}{\pi}$. Write an equation for the line tangent to the graph of g at $\left(\frac{1}{\pi}, -\frac{1}{\pi}\right)$ and use it to approximate $g\,(0.5)$.

$(\frac{-1}{\pi})^2 \cos 0$

$x \quad y$

$m = \frac{1}{\pi^2}\cdot 1$

$y - (-\frac{1}{\pi}) = \frac{1}{\pi^2}(x - \frac{1}{\pi})$

$\frac{\cancel{dx}}{\cancel{y^2}}\cdot\frac{dy}{\cancel{dx}} = \cancel{y^2}\cos(2-2\pi x)\cdot dx$

$\cancel{y^2}$

$$\int \frac{1}{y^2}\,dy = \int \cos(2-2\pi x)\,dx$$

$$\int y^{-2}\,dy = \int \cos(2-2\pi x)\,dx$$

$\frac{1}{-1}\cdot y^{-2+1=-1} + C_1$

$u = 2-2\pi x$

$du = -2\pi\,dx$

$\frac{du}{-2\pi} = dx$

$\frac{1}{y} = \frac{\sin(2-2\pi x)}{2\pi} - \pi$

$$\int \cos u\cdot\frac{du}{-2\pi}$$

$-y^{-1} = \frac{-1}{2\pi}\int \cos u\,du$

$-1\cdot\ -\frac{1}{y} = \left(\frac{-1}{2\pi}\sin u + C\right)\cdot -1$

$\frac{1}{y} = \frac{1}{2\pi}\sin(2-2\pi x) + C$

$\frac{1}{-\frac{1}{\pi}} = \frac{1}{2\pi}\sin(2 - 2\cancel{\pi}\cdot\frac{1}{\cancel{\pi}}) + C$

$-\pi = \frac{1}{2\pi}\sin(0) + C$

$-\pi = \frac{1}{2\pi}\cdot\sin 0 + C = \frac{1}{2\pi}\cdot 0 + C$

$\boxed{-\pi = C}$

(b) Find the particular solution to $y = g(x)$ to the given differential equation with the initial condition $g\left(\frac{1}{\pi}\right) = -\frac{1}{\pi}$.

$$\cancel{y} \cdot \frac{1}{\cancel{y}} = \left(\frac{1}{2\pi} \sin(2-2\pi x) - \pi\right) \cdot y$$

$$\frac{1}{\left(\frac{1}{2\pi} \sin(2-2\pi x) - \pi\right)} = y$$

c) Is $g'(x)$ increasing or decreasing at $x = \frac{1}{\pi}$?

$$\left(\frac{1}{\pi}, -\frac{1}{\pi}\right)$$

is The Slope of $g'(x)$ at $x = \frac{1}{\pi}$ + or —

$$g'(x) = \underbrace{y^2}_{}\underbrace{\cos(2-2\pi x)}_{}$$

$g''(x) =$ slope of $g'(x)$

$$g''(x) = 2y\frac{dy}{dx} \cdot \cos(2-2\pi x) + y^2 \cdot -\sin(2-2\pi x) \cdot -2\pi$$

$$g''\left(\frac{1}{\pi}\right) = 2y\left(\frac{1}{\pi^2}\right) \cdot \cos\left(2-2\pi \cdot \frac{1}{\pi}\right) + y^2 \cdot -\sin\left(2-2\pi \cdot \frac{1}{\pi}\right) \cdot -2\pi$$

$$g''\left(\frac{1}{\pi}\right) = 2 \cdot -\frac{1}{\pi} \cdot \frac{1}{\pi^2} \cdot \underset{1}{\cos 0} + \cancel{\left(\frac{-1}{\pi}\right)^2 \cdot -\underset{0}{\sin 0} \cdot -2\pi}$$

$g''\left(\frac{1}{\pi}\right) = -$ decreasing

$$-\ln\frac{\sqrt{2}}{2} = -\left[\ln\sqrt{2} - \ln 2\right] = -\ln\sqrt{2} + \ln 2 = \ln\frac{2}{\sqrt{2}} =$$

$$\ln\sqrt{2}$$

$$\ln a^b = b\ln a$$

$$\frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\cancel{2}\sqrt{2}}{\cancel{2}}$$

$$\frac{2}{\sqrt{2}} = \frac{\sqrt{2} \cdot \sqrt{2}}{\sqrt{2}}$$

11. (1969 AB29)

$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x}\,dx =$$

$u=\sin x$ $du=\cos x\,dx$ $\frac{du}{\cos x}=dx$

$\int \frac{\cancel{\cos x}}{u}\cdot\frac{du}{\cancel{\cos x}}=\int\frac{1}{u}du$

$\ln|u|+c$

$\ln|\sin x|+c \Big[_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

[✓ (A) $\ln\sqrt{2}$] (B) $\ln\frac{\pi}{4}$ (C) $\ln\sqrt{3}$ (D) $\ln\frac{\sqrt{3}}{2}$ (E) $\ln e$

$\ln|\sin\frac{\pi}{2}|+c-[\ln|\sin\frac{\pi}{4}|+c]$

$\ln|1|+\cancel{c}-\ln\frac{\sqrt{2}}{2}\cancel{-c}=0-\ln\frac{\sqrt{2}}{2}=\ln\left(\frac{\sqrt{2}}{2}\right)^{-1}=\ln\frac{2}{\sqrt{2}}=\ln\sqrt{2}$

$\ln$

1. (1973 AB 27)

$$\int_0^{1/2} \frac{2x}{\sqrt{1-x^2}}\,dx=$$

$u=1-x^2$ $du=-2x\,dx \Rightarrow \frac{du}{-2x}=dx$

(A) $1-\frac{\sqrt{3}}{2}$ (B) $\frac{1}{2}\ln\frac{3}{4}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{6}-1$ [✓ (E) $2-\sqrt{3}$]

$\int\frac{\cancel{2x}}{\sqrt{u}}\cdot\frac{du}{-\cancel{2x}}=-\int u^{-\frac{1}{2}}du=-\frac{2}{1}u^{-\frac{1}{2}+1=\frac{1}{2}}$

$-2\sqrt{1-x^2}\Big[_0^{\frac{1}{2}}=-2\sqrt{1-(\frac{1}{2})^2}-(-2\sqrt{1-0^2})$

$-2\sqrt{1-\frac{1}{4}}+2\sqrt{1}$

$-2\sqrt{\frac{3}{4}}+2=2-\cancel{2}\frac{\sqrt{3}}{\cancel{2}}$

If $\int_0^k \left(2kx-x^2\right)dx=18$, then $k=$

(A) -9 (B) -3 [✓ (C) 3] (D) 9 (E) 18

$\int(2kx-x^2)\,dx=2k\cdot\frac{1}{2}\cdot x^{1+1=2}-\frac{1}{3}x^{2+1=3}+c\Big[_0^k$

$k\cdot x^2-\frac{1}{3}x^3+c\Big[_0^k=k\cdot k^2-\frac{1}{3}\cdot k^3\cancel{+c}-\left[\cancel{k(0)^2-\frac{1}{3}(0)^3+c}\right]$

$1k^3-\frac{1}{3}k^3=\frac{2}{3}k^3=18$

$k^3=\frac{18\cdot 3}{2}=27$ $k^3=27$ $k=3$

2. (1985 BC3-appropriate for AB)

$1^2+2(1)=3$ $2^2+2^2=8$

$u=x^2+2x$

$du=(2x+2)dx \Rightarrow \frac{du}{2x+2}=dx$

$$\int_1^2 \frac{x+1}{x^2+2x}dx =$$

(A) $\ln 8-\ln 3$ [✓ (B) $\frac{\ln 8-\ln 3}{2}$] (C) $\ln 8$ (D) $\frac{3\ln 2}{2}$ (E) $\frac{3\ln 2+2}{2}$

$$\int \frac{x+1}{u}\cdot\frac{du}{2x+2} = \int \frac{\cancel{x+1}}{u}\,\frac{du}{2(\cancel{x+1})} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + c \Big[_3^8$$

$\frac{1}{2}\ln u + c \Big[_3^8$

$\frac{1}{2}\ln 8 - \frac{1}{2}\ln 3$

$\frac{1}{2}\ln|x^2+2x| + c \Big[_1^2$

$\frac{1}{2}\ln|2^2+2^2| - \frac{1}{2}\ln|1^2+2(1)|$

$\boxed{\frac{1}{2}\ln 8 - \frac{1}{2}\ln 3}$ = [illegible]

---

6. (1973 AB30)

$$\int_1^2 \frac{x-4}{x^2}dx = \int_1^2 \left[\frac{x}{x^2} - \frac{4}{x^2}\right]dx = \int_1^2 \left(\frac{1}{x} - 4x^{-2}\right)dx = \ln|x| - 4\cdot\frac{1}{-1}\cdot x^{-2+1=-1} + c$$

(A) $-\frac{1}{2}$ [✓ (B) $\ln 2-2$] (C) $\ln 2$ (D) 2 (E) $\ln 2+2$

$\ln|x| + 4x^{-1} + c \Big[_1^2$

$\ln|x| + \frac{4}{x} + c \Big[_1^2$

$\ln 2 + \frac{4}{2} - \left[\underset{0}{\ln 1} + \frac{4}{1}\right]$

$\ln 2 + 2 - 4 = \ln 2 \;-2$

8. (2012 AB12)

$u = \sqrt{x} = x^{\frac{1}{2}}$

$du = \frac{1}{2} x^{\frac{1}{2}-1} dx = \frac{1}{2} x^{-\frac{1}{2}} dx$

Using the substitution $u = \sqrt{x}$, $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ is equal to which of the following?

(A) $2\int_1^{16} e^u\,du$ (B) $2\int_1^{4} e^u\,du$ (C) $2\int_1^{2} e^u\,du$ (D) $\frac{1}{2}\int_1^{2} e^u\,du$ (E) $\int_1^{4} e^u\,du$

$\int_1^2 \frac{e^u}{\cancel{\sqrt{x}}} \cdot 2\cancel{\sqrt{x}}\,du = 2\int e^u du$

$du = \frac{1}{2\sqrt{x}} dx$

$2e^u + c = 2e^{\sqrt{x}} + c \Big|_1^4$

$2\sqrt{x}\,du = dx$

$u = \sqrt{x}$

$u = \sqrt{4} = 2$

$u = \sqrt{1} = 1$

15. (1997 BC1-appropriate for AB)

$\int_0^1 \sqrt{x}(x+1)\,dx = = \int_0^1 x^{\frac{1}{2}}(x+1)\,dx = \int_0^1 [x^{\frac{3}{2}} + x^{\frac{1}{2}}]\,dx = \frac{2}{5} \cdot x^{\frac{3}{2}+1=\frac{5}{2}} + \frac{2}{3}x^{\frac{1}{2}+1=\frac{3}{2}}$

(A) 0 (B) 1 [✓ (C) $\frac{16}{15}$] (D) $\frac{7}{5}$ (E) 2

$\frac{2}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}} + c \Big|_0^1$

$\frac{3 \cdot 2}{3 \cdot 5} + \frac{2 \cdot 5}{3 \cdot 5} - 0 - 0$

$\frac{6}{15} + \frac{10}{15} = \frac{16}{15} = 1\frac{1}{15}$

5. (1985 BC40-appropriate for AB)

$u = \frac{x}{2} \Leftrightarrow 2u = x$

$du = \frac{1}{2}dx$

$2du = dx$

If the substitution $u = \frac{x}{2}$ is made, the integral $\int_2^4 \frac{1-\left(\frac{x}{2}\right)^2}{x}dx =$

(A) $\int_1^2 \frac{1-u^2}{u}du$ (B) $\int_2^4 \frac{1-u^2}{u}du$ (C) $\int_1^2 \frac{1-u^2}{2u}du$

(D) $\int_1^2 \frac{1-u^2}{4u}du$ (E) $\int_2^4 \frac{1-u^2}{2u}du$

$x = 4$ $u = \frac{x}{2}$

$u = \frac{4}{2} = 2$

$\int_1^2 \frac{1-u^2}{2u} \cdot 2du$

$x = 2$ $u = \frac{x}{2}$

$u = \frac{2}{2} = 1$

---

4. (1973 BC38-appropriate for AB)

$u = x - c$

$du = dx$

$x = 1$ $u = 1 - c$

$x = 2$ $u = 2 - c$

If $\int_1^2 f(x-c)\,dx = 5$ where c is a constant, then $\int_{1-c}^{2-c} f(x)\,dx =$

(A) $5+c$ [✓ (B) 5] (C) $5-c$ (D) $c-5$ (E) -5

$\int_{1-c}^{2-c} F(u) = 5$

$\int_{1-c}^{2-c} F(x)\,dx$

$\int_1^2 F(x-c)\,dx = 5$

$\uparrow$

Start

$x - c = 0$

$x = c$

Start

$x = 0$

$1 - c$

12. (1969 AB38)

$\int \frac{x^2}{e^{x^3}}\,dx =$

$u = x^3$

$du = 3x^2\,dx$

$\frac{du}{3x^2} = dx$

(A) $-\frac{1}{3}\ln e^{x^3} + C$

(B) $-\frac{e^{x^3}}{3} + C$

[✓ (C) $-\frac{1}{3e^{x^3}} + C$]

(D) $\frac{1}{3}\ln e^{x^3} + C$

(E) $\frac{x^3}{3e^{x^3}} + C$

$$\int \frac{\cancel{x^2}}{e^u} \cdot \frac{dU}{3\cancel{x^2}} = \frac{1}{3}\int \frac{du}{e^u} = \frac{1}{3}\int e^{-u}\,du = \frac{1}{3}\int e^{L} \cdot -dL = -\frac{1}{3}\int e^{L}\,dL$$

$L = -u$

$dL = -du$

$-dL = du$

$-\frac{1}{3}e^{L} + c$

$-\frac{1}{3}e^{-u} + c$

$-\frac{1}{3}e^{-x^3} + c$

$\frac{-1}{3e^{x^3}} + C$

1. On a certain workday, the rate, in tons per hour, at which unprocessed gravel arrives at a gravel processing plant is modeled by $G(t) = 90 + 45\cos\left(\frac{t^2}{18}\right)$, where t is measured in hours and $0 \le t \le 8$. At the beginning of the workday $(t = 0)$, the plant has 500 tons of unprocessed gravel. During the hours of operation, $0 \le t \le 8$, the plant processes gravel at a constant rate of 100 tons per hour.

in

$T=0$

out

(d) What is the maximum amount of unprocessed gravel at the plant during the hours of operation on this workday? Justify your answer.

$$\boxed{0 = 90 + 45\cos\left(\frac{T^2}{18}\right) - 100}$$

$T = 4.92348$

$h(x)$ = amount of unprocessed gravel

$h'(x) = 0$ = Rate of unprocessed gravel

$y = 90 + 45\cos\left(\frac{x^2}{18}\right) - 100$

$T = 0$

$T = 8$

$T = 4.92348$

(d) The amount of unprocessed gravel at time t is given by $A(t) = \underline{500} + \int_0^t (G(s) - 100)\,ds.$

$$A'(t) = G(t) - 100 = 0 \Rightarrow t = 4.9\underline{23480}$$

t	$A(t)$
0	500 ✓
4.92348	635.376123
8	525.551089

The maximum amount of unprocessed gravel at the plant during this workday is 635.376 tons.

$$3 : \begin{cases} 1 : \text{considers } A'(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$$